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THE ORBIT DECOMPOSITION AND ORBIT TYPE OF THE AUTOMORPHISM GROUP OF CERTAIN EXCEPTIONAL JORDAN ALGEBRA AND ITS APPLICATIONS

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ABSTRACT. Let \mathcal{J}^1 be the real form of complex simple Jordan algebra with the automorphism group $F_{4(-20)}$. The classification of $F_{4(-20)}$ -orbits and the stabilizer groups of $F_{4(-20)}$ -orbit on \mathcal{J}^1 are determined. As applications, for $F_{4(-20)}$, the Bruhat and Gauss decomposition, the Iwasawa decomposition and also the Iwasawa decomposition with respect to K_e in sense of T. Oshima and J. Sekiguchi are given concretely.

1. THE EXCEPTIONAL JORDAN ALGEBRA \mathcal{J}^1 AND THE AUTOMORPHISM GROUP $F_{4(-20)}$.

Denote the cartesian n -power of a set X as $X^n := X \times \cdots \times X$ (n times). For $\mathbb{F} = \mathbb{R}$ or \mathbb{C} , let V be a \mathbb{F} -linear space, $\mathrm{GL}_{\mathbb{F}}(V)$ the group of \mathbb{F} -linear automorphism of V , and $\mathrm{End}_{\mathbb{F}}(V)$ the linear space of \mathbb{F} -linear endomorphisms on V . A subset C is said to be a *cone* if $x \in V$ and $\lambda > 0$ imply that $\lambda x \in C$. For a mapping $f : V \rightarrow V$ and $c \in \mathbb{F}$, put $V_{f,c} := \{v \in V \mid f(v) = cv\}$ and $V_f := V_{f,1}$. Let G be a subgroup of $\mathrm{GL}_{\mathbb{F}}(V)$, ϕ an automorphism on G and $v, v_i \in V$. Then denote the subgroups $G^\phi := \{g \in G \mid \phi g = g\}$, the *stabilizer* of v as $G_v := \{g \in G \mid gv = v\}$ and $G_{v_1, \dots, v_n} := \bigcap_{i=1}^n G_{v_i}$. And denote the G -orbit of v as $\mathrm{Orb}_G(v) := \{gv \mid g \in G\}$.

For \mathbf{H} (Quaternions), the \mathbf{O} (*Octonions*) is defined as $\mathbf{O} := \mathbf{H} \oplus \mathbf{H}e = \{m + ae \mid m, a \in \mathbf{H}\}$, the *conjugation*, the *multiplication*, the *inner product* and the *quadratic form* as $\overline{m + ae} := \overline{m} - ae$, $(m + ae)(n + be) := (mn - \overline{b}a) + (a\overline{n} + bm)e$ (especially, $e^2 = -1$), $(m + ae|n + be) := (m|n) + (a|b)$ and $n(x) := (x|x)$, respectively. For $x \in \mathbf{O}$, the *scalar part* and the *vector part* of x and the set $\mathrm{Im}\mathbf{O}$ are defined by $\mathrm{Re}(x) := \frac{1}{2}(x + \overline{x})$, $\mathrm{Im}(x) := \frac{1}{2}(x - \overline{x})$ and $\mathrm{Im}\mathbf{O} := \{x \in \mathbf{O} \mid \overline{x} = -x\}$, respectively.

For $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ and $x = (x_1, x_2, x_3) \in \mathbf{O}^3$, denote

$$h^1(\xi; x) := \begin{pmatrix} \xi_1 & \sqrt{-1}x_3 & \sqrt{-1}\overline{x_2} \\ \sqrt{-1}\overline{x_3} & \xi_2 & x_1 \\ \sqrt{-1}x_2 & \overline{x_1} & \xi_3 \end{pmatrix}$$

and

$$\mathcal{J}^1 := \{h^1(\xi; x) \mid \xi \in \mathbb{R}^3, x \in \mathbf{O}^3\}.$$

The *Jordan product* is defined by

$$X \circ Y := \frac{1}{2}(XY + YX) \quad \text{for } X, Y \in \mathcal{J}^1.$$

Then the identity element of the Jordan product is $E := \text{diag}(1, 1, 1)$. For $X = h^1(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3)$ and $Y = h^1(\eta_1, \eta_2, \eta_3; y_1, y_2, y_3) \in \mathcal{J}^1$, the *trace* and the *inner product* are defined as

$$\text{tr}(X) = \xi_1 + \xi_2 + \xi_3,$$

$$(X|Y) = \text{tr}(X \circ Y) = \left(\sum_{k=1}^3 \xi_k \eta_k\right) + 2(x_1|y_1) - 2(x_2|y_2) - 2(x_3|y_3),$$

respectively, the *cross product* and the *determinant* as

$$X \times Y := \frac{1}{2}(2X \circ Y - \text{tr}(X)Y - \text{tr}(Y)X + (\text{tr}(X)\text{tr}(Y) - (X|Y))E),$$

$$\begin{aligned} \det(X) &:= \frac{1}{3}(X|X \times X) \\ &= \xi_1 \xi_2 \xi_3 + 2\text{Re}((x_1 x_2) x_3) - \xi_1(x_1|x_1) + \xi_2(x_2|x_2) + \xi_3(x_3|x_3) \end{aligned}$$

respectively. Hereafter we denote $X \times X := X^{\times 2}$. The *characteristic polynomial* $\Phi_X(\lambda)$ of $X \in \mathcal{J}^1$ is defined by

$$\begin{aligned} \Phi_X(\lambda) &:= \det(\lambda E - X) = \frac{1}{3}(\lambda E - X|(\lambda E - X)^{\times 2}) \\ &= \lambda^3 - \text{tr}(X)\lambda^2 + \text{tr}(X^{\times 2})\lambda - \det(X). \end{aligned}$$

For $i \in \{1, 2, 3\}$ and $x \in \mathbf{O}$, denote

$$\begin{aligned} E_i &:= h^1(\delta_{i1}, \delta_{i2}, \delta_{i3}; 0, 0, 0), & F_i^1(x) &:= h^1(0, 0, 0; \delta_{i1}x, \delta_{i2}x, \delta_{i3}x), \\ P^+ &:= h^1(1, -1, 0; 0, 0, 1), & P^- &:= h^1(-1, 1, 0; 0, 0, 1), \\ Q^+(x) &:= h^1(0, 0, 0; x, \bar{x}, 0), & Q^-(x) &:= h^1(0, 0, 0; x, -\bar{x}, 0) \end{aligned}$$

where δ_{ij} is the Kronecker's delta. Then $X \in \mathcal{J}^1$ can be expressed by

$$X = h^1(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3) = \sum_{i=1}^3 (\xi_i E_i + F_i^1(x_i))$$

for some $\xi_i \in \mathbb{R}$ and $x_i \in \mathbf{O}$, and denote

$$(X)_{E_i} := \xi_i = (X|E_i), \quad (X)_{F_i^1} := x_i.$$

Lemma 1.1. (cf. [25, Lemma 1.6 with $\mathcal{J}^1 \subset \mathcal{J}^{\mathbb{C}}$]) *For all $X \in \mathcal{J}^1$,*

$$(X^{\times 2})^{\times 2} = \det(X)X.$$

The linear Lie group $F_{4(-20)}$ is defined by

$$F_{4(-20)} := \text{Aut}(\mathcal{J}^1) = \{g \in \text{GL}_{\mathbb{R}}(\mathcal{J}^1) \mid g(X \circ Y) = gX \circ gY\}.$$

The following result is proved after [34, 35], [39, Lemma 2.1.2, Proposition 2.1.3] and [33, p.159, Proposition 5.9.4, §5.10].

Proposition 1.2. (cf. [24, Theorem 1.4], [25, Proposition 0.1(1)])

$$\begin{aligned} F_{4(-20)} &= \{g \in F_{4(-20)} \mid \text{tr}(gX) = \text{tr}(X)\} \\ &= \{g \in \text{GL}_{\mathbb{R}}(\mathcal{J}^1) \mid \det(gX) = \det(X), \ gE = E\} \\ &= \{g \in \text{GL}_{\mathbb{R}}(\mathcal{J}^1) \mid \Phi_{gX}(\lambda) = \Phi_X(\lambda)\} \\ &= \{g \in \text{GL}_{\mathbb{R}}(\mathcal{J}^1) \mid \det(gX) = \det(X), \ (gX|gY) = (X|Y)\} \\ &= \{g \in \text{GL}_{\mathbb{R}}(\mathcal{J}^1) \mid g(X \times Y) = gX \times gY\}. \end{aligned}$$

A *characteristic root* of $X \in \mathcal{J}^1$ is said to be a solution of $\Phi_X(\lambda) = 0$ over \mathbb{C} . By Proposition 1.2, the trace, the inner product, the determinant, the identity element, the cross product and the characteristic polynomial are invariant under the action of $F_{4(-20)}$. Moreover the set of all characteristic roots and those multiplicities are invariant under the action of $F_{4(-20)}$.

Proposition 1.3. ([39]) $F_{4(-20)}$ is a connected and simply connected non-compact simple real Lie group of type $\mathbf{F}_{4(-20)}$.

2. THE ORBIT DECOMPOSITION OF $F_{4(-20)}$ -ORBITS ON \mathcal{J}^1 .

The subset $\mathcal{H} \subset \mathcal{J}^1$ and the *Cayley hyperbolic planes* $\mathcal{H}(\mathbf{O})$ and $\mathcal{H}'(\mathbf{O})$ of \mathcal{J}^1 are defined as

$$\begin{aligned} \mathcal{H} &:= \{X \in \mathcal{J}^1 \mid X^{\times 2} = 0, \ \text{tr}(X) = 1\}, \\ \mathcal{H}(\mathbf{O}) &:= \{X \in \mathcal{J}^1 \mid X^{\times 2} = 0, \ \text{tr}(X) = 1, \ (X|E_1) \geq 1\}, \\ \mathcal{H}'(\mathbf{O}) &:= \{X \in \mathcal{J}^1 \mid X^{\times 2} = 0, \ \text{tr}(X) = 1, \ (X|E_1) \leq 0\} \end{aligned}$$

respectively.

Proposition 2.1. (cf. [24, Propositions 1.6(1) and 2.10])

- (1) $\mathcal{H} = \mathcal{H}(\mathbf{O}) \amalg \mathcal{H}'(\mathbf{O})$.
- (2) $\mathcal{H}(\mathbf{O}) = \text{Orb}_{F_{4(-20)}}(E_1)$.
- (3) $\mathcal{H}'(\mathbf{O}) = \text{Orb}_{F_{4(-20)}}(E_2) = \text{Orb}_{F_{4(-20)}}(E_3)$.

The cone \mathcal{N} of \mathcal{J}^1 is defined by

$$\mathcal{N} = \{X \in \mathcal{J}^1 \mid \text{tr}(X) = \text{tr}(X^{\times 2}) = \det(X) = 0\}.$$

Then using Lemma 1.1, \mathcal{N} contains the following cones:

$$\begin{aligned}\mathcal{N}_1(\mathbf{O}) &:= \{X \in \mathcal{J}^1 \mid X^{\times 2} = 0, \operatorname{tr}(X) = 0, X \neq 0\}, \\ \mathcal{N}_1^+(\mathbf{O}) &:= \{X \in \mathcal{J}^1 \mid X^{\times 2} = 0, \operatorname{tr}(X) = 0, (X|E_1) > 0\}, \\ \mathcal{N}_1^-(\mathbf{O}) &:= \{X \in \mathcal{J}^1 \mid X^{\times 2} = 0, \operatorname{tr}(X) = 0, (X|E_1) < 0\}, \\ \mathcal{N}_2(\mathbf{O}) &:= \{X \in \mathcal{J}^1 \mid \operatorname{tr}(X) = \operatorname{tr}(X^{\times 2}) = \det(X) = 0, X^{\times 2} \neq 0\}, \\ \mathcal{N}_0(\mathbf{O}) &:= \{0\}.\end{aligned}$$

Proposition 2.2. (cf. [24, Propositions 1.6(2), 2.10(2) and 4.3(4)])

- (1) $\mathcal{N}_1(\mathbf{O}) = \mathcal{N}_1^+(\mathbf{O}) \amalg \mathcal{N}_1^-(\mathbf{O})$.
- (2) $\mathcal{N} = \mathcal{N}_0(\mathbf{O}) \amalg \mathcal{N}_1^+(\mathbf{O}) \amalg \mathcal{N}_1^-(\mathbf{O}) \amalg \mathcal{N}_2(\mathbf{O})$.
- (3) $\mathcal{N}_1^+(\mathbf{O}) = \operatorname{Orb}_{F_4(-20)}(P^+)$.
- (4) $\mathcal{N}_1^-(\mathbf{O}) = \operatorname{Orb}_{F_4(-20)}(P^-)$.
- (5) $\mathcal{N}_2(\mathbf{O}) = \operatorname{Orb}_{F_4(-20)}(Q^+(1))$.

For $X \in \mathcal{J}^1$, denote $L^\times(X) \in \operatorname{End}_{\mathbb{R}}(\mathcal{J}^1)$ as

$$L^\times(X)Y := X \times Y \quad \text{for } Y \in \mathcal{J}^1$$

and the *minimal space* of X as

$$V_X := \{aX^{\times 2} + bX + cE \mid a, b, c \in \mathbb{R}\}.$$

Then V_X is closed under the cross product ([25, Lemma 1.6(3)]). And for $\lambda_0 \in \mathbb{R}$, denote the elements $p(X)$, E_{X,λ_0} , $W_{X,\lambda_0} \in V_X$ as

$$\begin{aligned}p(X) &:= X - \frac{1}{3}\operatorname{tr}(X)E, \\ E_{X,\lambda_0} &:= \frac{1}{\operatorname{tr}((\lambda_0 E - X)^{\times 2})}(\lambda_0 E - X)^{\times 2}, \\ W_{X,\lambda_0} &:= X - (\lambda_0 E_{X,\lambda_0} + \frac{\operatorname{tr}(X) - \lambda_0}{2}(E - E_{X,\lambda_0}))\end{aligned}$$

respectively. If E_{X,λ_1} is well-defined (ie, $\operatorname{tr}((\lambda_1 E - X)^{\times 2}) \neq 0$), then

$$X = \lambda_0 E_{X,\lambda_0} + \frac{\operatorname{tr}(X) - \lambda_0}{2}(E - E_{X,\lambda_0}) + W_{X,\lambda_0}.$$

For $r \in \mathbb{R}$, consider the eigenspace $\mathcal{J}_{L^\times(2E_{X,\lambda_1}),r}^1$. Then we have the following two lemmas (cf. [24]):

Lemma 2.3. Let $X \in \mathcal{J}^1$. Then for all $g \in F_4(-20)$,

$$g(V_X) = V_{gX}, \quad gE_{X,\lambda_1} = E_{gX,\lambda_1}, \quad gW_{X,\lambda_1} = W_{gX,\lambda_1}, \quad gp(X) = p(gX).$$

Lemma 2.4. Assume that $X \in \mathcal{J}^1$ has a characteristic root $\lambda_1 \in \mathbb{R}$ of multiplicity 1.

(1) E_{X,λ_1} is well-defined (ie, $\operatorname{tr}((\lambda_1 E - X)^{\times 2}) \neq 0$), and $E_{X,\lambda_1} \in \mathcal{H} \cap V_X$.

(2) $E_{X,\lambda_1} \in \mathcal{J}_{L^\times(2E_{X,\lambda_1}),0}^1$, $E - E_{X,\lambda_1} \in \mathcal{J}_{L^\times(2E_{X,\lambda_1}),1}^1 \cap V_X$ and $W_{X,\lambda_1} \in \mathcal{J}_{L^\times(2E_{X,\lambda_1}),-1}^1 \cap V_X$.

Main Theorem 1. ($F_{4(-20)}$ -orbits on \mathcal{J}^1 [24, Main Theorem])

$F_{4(-20)}$ -orbits on \mathcal{J}^1 are classified as follows.

(I) Assume that $X \in \mathcal{J}^1$ admits the characteristic roots $\lambda_1 > \lambda_2 > \lambda_3$. Then there exists the unique $i \in \{1, 2, 3\}$ such that $\mathcal{H}(\mathbf{O}) \cap V_X = \{E_{X,\lambda_i}\}$ and $\mathcal{H}'(\mathbf{O}) \cap V_X = \{E_{X,\lambda_{i+1}}, E_{X,\lambda_{i+2}}\}$ where $i, i+1, i+2$ are counted modulo 3. In this case, X can be transformed to one of the following canonical forms by $F_{4(-20)}$.

Cases	The canonical forms of X
1. $E_{X,\lambda_1} \in \mathcal{H}(\mathbf{O})$	$\text{diag}(\lambda_1, \lambda_2, \lambda_3)$
2. $E_{X,\lambda_2} \in \mathcal{H}(\mathbf{O})$	$\text{diag}(\lambda_2, \lambda_3, \lambda_1)$
3. $E_{X,\lambda_3} \in \mathcal{H}(\mathbf{O})$	$\text{diag}(\lambda_3, \lambda_1, \lambda_2)$

(II) Assume that $X \in \mathcal{J}^1$ admits the characteristic roots $\lambda_1 \in \mathbb{R}$, $p \pm \sqrt{-1}q$ with $p \in \mathbb{R}$ and $q > 0$. Then X can be transformed to the following canonical form by $F_{4(-20)}$.

the characteristic roots of X	The canonical form of X
4. $\lambda_1 \in \mathbb{R}, p \pm \sqrt{-1}q$	$\text{diag}(p, p, \lambda_1) + F_3^1(q)$

(III) Assume that $X \in \mathcal{J}^1$ admits the characteristic roots λ_1 of multiplicity 1 and λ_2 of multiplicity 2. Then $W_{X,\lambda_1} \in \mathcal{N}_1(\mathbf{O}) \amalg \{0\}$. In this case, X can be transformed to one of the following canonical forms by $F_{4(-20)}$.

Cases	The canonical form of X
5. $E_{X,\lambda_1} \in \mathcal{H}(\mathbf{O})$	$\text{diag}(\lambda_1, \lambda_2, \lambda_2)$
6. $E_{X,\lambda_1} \in \mathcal{H}'(\mathbf{O}), W_{X,\lambda_1} = 0$	$\text{diag}(\lambda_2, \lambda_2, \lambda_1)$
7. $W_{X,\lambda_1} \in \mathcal{N}_1^+(\mathbf{O})$	$\text{diag}(\lambda_2, \lambda_2, \lambda_1) + P^+$
8. $W_{X,\lambda_1} \in \mathcal{N}_1^-(\mathbf{O})$	$\text{diag}(\lambda_2, \lambda_2, \lambda_1) + P^-$

(IV) Assume that $X \in \mathcal{J}^1$ admits the characteristic root of multiplicity 3. Then $p(X) \in \mathcal{N}$. In this case, X can be transformed to one of the following canonical forms by $F_{4(-20)}$.

Cases	The canonical form of X
9. $p(X) = 0$	$\frac{1}{3}\text{tr}(X)E$
10. $p(X) \in \mathcal{N}_1^+(\mathbf{O})$	$\frac{1}{3}\text{tr}(X)E + P^+$
11. $p(X) \in \mathcal{N}_1^-(\mathbf{O})$	$\frac{1}{3}\text{tr}(X)E + P^-$
12. $p(X) \in \mathcal{N}_2(\mathbf{O})$	$\frac{1}{3}\text{tr}(X)E + Q^+(1)$

(V) By $F_{4(-20)}$, the above canonical forms cannot be transformed from each other.

3. THE STABILIZER GROUPS OF SPIN GROUP TYPE.

Let G be a topological group with identity element 1. Then G^0 denotes the identity connected component. Denote the quadratic form $Q_{p,q}$ on \mathbb{R}^{p+q} as $Q_{p,q}(x) := -(x_1^2 + \cdots + x_p^2) + (x_{p+1}^2 + \cdots + x_{p+q}^2)$ for $x = (x_1, \dots, x_{p+q})$, the quadratic space as $(\mathbb{R}^{p,q}, Q_{p,q})$, the set of all orthogonal transformations as $O(\mathbb{R}^{p,q}, Q_{p,q})$ and $SO(\mathbb{R}^{p,q}, Q_{p,q}) := \{g \in O(\mathbb{R}^{p,q}, Q_{p,q}) \mid \det(g) = 1\}$ where $\det(g)$ is the determinant of $g \in \text{End}_{\mathbb{R}}(\mathbb{R}^{p,q})$. Then $O(\mathbb{R}^{p,q}, Q_{p,q})$ and $SO(\mathbb{R}^{p,q}, Q_{p,q})$ are linear Lie groups. Denote the quadratic form Q on \mathcal{J}^1 as $Q(X) := -\text{tr}(X^{\times 2})$ for $X \in \mathcal{J}^1$ and consider the subspace $\mathcal{J}_{0,9}^1$, $\mathcal{J}_{8,1}^1$ and $\mathcal{J}_{7,1}^1$ of eigenspace of $L^{\times}(2E_i)$ with eigenvalue -1 as

$$\begin{aligned}\mathcal{J}_{0,9}^1 &:= \mathcal{J}_{L^{\times}(2E_1), -1}^1, & \mathcal{J}_{8,1}^1 &:= \mathcal{J}_{L^{\times}(2E_3), -1}^1, \\ \mathcal{J}_{7,1}^1 &:= \{X \in \mathcal{J}_{8,1}^1 \mid (F_3^1(1)|X) = 0\}.\end{aligned}$$

Then $\mathcal{J}_{0,9}^1 = \{\xi(E_2 - E_3) + F_1^1(x) \mid \xi \in \mathbb{R}, x \in \mathbf{O}\}$, $\mathcal{J}_{8,1}^1 = \{\xi(E_1 - E_2) + F_3^1(x) \mid \xi \in \mathbb{R}, x \in \mathbf{O}\}$ and $\mathcal{J}_{7,1}^1 = \{\xi(E_1 - E_2) + F_3^1(x) \mid \xi \in \mathbb{R}, x \in \text{Im } \mathbf{O}\}$. Since $Q(\xi(E_2 - E_3) + F_1^1(x)) = \xi^2 + n(x)$ and $Q(\xi(E_1 - E_2) + F_3^1(x)) = \xi^2 - n(x)$, we see that $(\mathcal{J}_{0,9}^1, Q)$, $(\mathcal{J}_{8,1}^1, Q)$ and $(\mathcal{J}_{7,1}^1, Q)$ are isomorphic to $(\mathbb{R}^9, |\cdot|^9)$, $(\mathbb{R}^{8,1}, Q_{8,1})$ and $(\mathbb{R}^{7,1}, Q_{7,1})$, respectively. Moreover, denote

$$\begin{aligned}S^8 &:= \{X \in \mathcal{J}_{0,9}^1 \mid Q(X) = 1\}, \\ S_+^{8,1} &:= \{X \in \mathcal{J}_{8,1}^1 \mid Q(X) = 1, (E_3|X) > 0\}, \\ S_+^{7,1} &:= \{X \in \mathcal{J}_{7,1}^1 \mid Q(X) = 1, (E_3|X) > 0\}.\end{aligned}$$

From now on, the groups $SO(8)$ and $SO(7)$ are identified with the groups $SO(8) = \{g \in \text{GL}_{\mathbb{R}}(\mathbf{O}) \mid (gx|gy) = (x|y), \det(g) = 1\}$ and $SO(7) = \{g \in SO(8) \mid g1 = 1\}$, respectively. The subgroup $T(\mathbf{O})$ of $SO(8)^3$ is defined as

$$T(\mathbf{O}) := \{(g_1, g_2, g_3) \in SO(8)^3 \mid (g_1x)(g_2y) = g_3(xy) \text{ for all } x, y \in \mathbf{O}\}$$

(cf. [2], [9, (2.4.6)], [22], [33], [43]), and the subgroup \tilde{D}_4 of $SO(8)^3$ as

$$\tilde{D}_4 := \{(g_1, g_2, g_3) \in SO(8)^3 \mid (g_1x)(g_2y) = \overline{g_3(\overline{xy})} \text{ for all } x, y \in \mathbf{O}\}.$$

For $i \in \{1, 2, 3\}$, the homomorphism $p_i : \tilde{D}_4 \rightarrow SO(8)$ is defined by

$$p_i(g_1, g_2, g_3) := g_i \quad \text{for } (g_1, g_2, g_3) \in \tilde{D}_4.$$

The subgroup \tilde{B}_3 of \tilde{D}_4 is defined as

$$\tilde{B}_3 := \{(g_1, g_2, g_3) \in \tilde{D}_4 \mid g_3 1 = 1\}$$

and the homomorphism $q : \tilde{B}_3 \rightarrow SO(7)$ as $q := p_3|_{\tilde{B}_3}$. Denote $\epsilon_i(j) := (-1)^{1+\delta_{ij}}$ where δ_{ij} is the Kronecker delta. Thus if $i = j$, then $\epsilon_i(j) = 1$, else $\epsilon_i(j) = -1$.

Lemma 3.1.

- (1) ([43, Theorems 1.15.1 and 1.16.1]) \tilde{D}_4 and \tilde{B}_3 are connected.
 (2) (The principle of triality: [2], [9, (2.4.6)], cf. [43, Theorem 1.14.2])
 The following sequence is exact:

$$1 \rightarrow \{(1, 1, 1), (\epsilon_i(1), \epsilon_i(2), \epsilon_i(3))\} \rightarrow \tilde{D}_4 \xrightarrow{P_i} \mathrm{SO}(8) \rightarrow 1.$$

- (3) ([43, Theorem 1.15.2])
 The following sequence is exact:

$$1 \rightarrow \{(1, 1, 1), (-1, -1, 1)\} \rightarrow \tilde{B}_3 \xrightarrow{q} \mathrm{SO}(7) \rightarrow 1.$$

By Lemma 3.1, we see that \tilde{D}_4 is connected and a two-fold covering group of $\mathrm{SO}(8)$, and \tilde{B}_3 is connected and a two-fold covering group of $\mathrm{SO}(7)$. So denote

$$\mathrm{Spin}(8) := \tilde{D}_4, \quad \mathrm{Spin}(7) := \tilde{B}_3.$$

Lemma 3.2. ([22], cf. [43, Theorem 2.7.1], [26, lemma 3.2])

The following homomorphisms are group isomorphisms:

$$(1) \varphi_0 : \mathrm{Spin}(8) \rightarrow (F_{4(-20)})_{E_1, E_2, E_3};$$

$$\varphi_0(g_1, g_2, g_3) \left(\sum (\xi_i E_i + F_i^1(x_i)) \right) = \sum (\xi_i E_i + F_i^1(g_i x_i)),$$

$$(2) \varphi_0 : \mathrm{Spin}(7) \rightarrow (F_{4(-20)})_{E_1, E_2, F_3^1(1)}; \quad \varphi_0 = \varphi_0|_{\mathrm{Spin}(7)}.$$

Hereafter $\mathrm{Spin}(8)$ and $\mathrm{Spin}(7)$ are identified with $(F_{4(-20)})_{E_1, E_2, E_3}$ and $(F_{4(-20)})_{E_1, E_2, F_3^1(1)}$ via φ_0 , respectively.

Lemma 3.3. ([38], [39], cf. [26, Lemmas 3.9 and 3.12])

$$(1) (F_{4(-20)})_{E_1} / \mathrm{Spin}(8) \simeq S^8, \quad (2) (F_{4(-20)})_{E_3} / \mathrm{Spin}(8) \simeq S_+^{8,1},$$

$$(3) (F_{4(-20)})_{F_3^1(1)} / \mathrm{Spin}(7) \simeq S_+^{7,1}.$$

Furthermore, $(F_{4(-20)})_{E_1}$, $(F_{4(-20)})_{E_3}$ and $(F_{4(-20)})_{F_3^1(1)}$ are connected.

Lemma 3.4. ([38], [39], cf. [26, Lemmas 3.10 and 3.13])

- (1) The following sequence is exact.

$$1 \rightarrow \mathbb{Z}_2 \rightarrow (F_{4(-20)})_{F_3^1(1)} \xrightarrow{f} \mathrm{O}^0(\mathcal{J}_{7,1}^1, Q) \rightarrow 1$$

where $f(g) = g|_{\mathcal{J}_{7,1}^1}$.

- (2) The following sequence is exact.

$$1 \rightarrow \mathbb{Z}_2 \rightarrow (F_{4(-20)})_{E_1} \xrightarrow{f} \mathrm{SO}(\mathcal{J}_{0,9}^1, Q) \rightarrow 1$$

where $f(g) = g|_{\mathcal{J}_{0,9}^1}$.

- (3) The following sequence is exact.

$$1 \rightarrow \mathbb{Z}_2 \rightarrow (F_{4(-20)})_{E_3} \xrightarrow{f} \mathrm{O}^0(\mathcal{J}_{8,1}^1, Q) \rightarrow 1$$

where $f(g) = g|_{\mathcal{J}_{8,1}^1}$.

Since Lemmas 3.3, 3.4 and $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2 = \pi_1(\mathrm{O}^0(n, 1))$ ($n \geq 3$), we can put

$$\mathrm{Spin}^0(7, 1) := (F_{4(-20)})_{F_3^1(1)}, \quad \mathrm{Spin}(9) := (F_{4(-20)})_{E_1},$$

$$\mathrm{Spin}^0(8, 1) := (F_{4(-20)})_{E_3} \cong (F_{4(-20)})_{E_2}.$$

The element $\sigma_i \in F_{4(-20)}$ is defined by

$$\sigma_i \left(\sum_{j=1}^3 (\xi_j E_j + F_j^1(x_j)) \right) := \sum_{j=1}^3 (\xi_j E_j + \epsilon_i(j) F_j^1(x_j))$$

[38] (cf. [39]) where indices are counted modulo 3. The involutive automorphism $\tilde{\sigma}_i$ of $F_{4(-20)}$ is defined as

$$\tilde{\sigma}_i(g) := \sigma_i g \sigma_i \quad \text{for } g \in F_{4(-20)},$$

and the subgroup K of $F_{4(-20)}$ as

$$K := (F_{4(-20)})^{\tilde{\sigma}_1} = \{g \in F_{4(-20)} \mid \sigma_1 g = g \sigma_1\}.$$

Proposition 3.5. ([38, Theorem 8], [39, Theorem 2.4.4], cf. [26, Proposition 3.16]).

- (1) $(F_{4(-20)})^{\tilde{\sigma}_i} = (F_{4(-20)})_{E_i}$.
- (2) $K = (F_{4(-20)})_{E_1} = \mathrm{Spin}(9)$.
- (3) $(F_{4(-20)})^{\tilde{\sigma}_2} = (F_{4(-20)})_{E_2} \cong \mathrm{Spin}^0(8, 1)$.

4. THE STABILIZER GROUPS OF SEMIDIRECT PRODUCT GROUP TYPE.

Denote the Lie algebras $\mathfrak{o}(8) = \mathrm{Lie}(\mathrm{O}(8))$ and $\mathfrak{f}_{4(-20)} = \mathrm{Lie}(F_{4(-20)})$. Since $\varphi_0 : D_4 \rightarrow (F_{4(-20)})_{E_1, E_2, E_3}$ is an isomorphism by Lemma 3.2, the Lie subalgebra \mathfrak{d}_4 of $\mathfrak{f}_{4(-20)}$ is defined by

$$\mathfrak{d}_4 := \left\{ d\varphi_0(D_1, D_2, D_3) \mid \begin{array}{l} (D_1, D_2, D_3) \in \mathfrak{o}(8)^3, \\ (D_1 x)y + x(D_2 y) = \overline{D_3(xy)} \\ \text{for all } x, y \in \mathbf{O} \end{array} \right\}.$$

Then

$$d\varphi_0(D_1, D_2, D_3) \left(\sum (\xi_i E_i + F_i^1(x_i)) \right) = \sum F_i^1(D_i x_i).$$

For $a \in \mathbf{O}$, denote

$$A_1^1(a) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -\bar{a} & 0 \end{pmatrix}, \quad A_2^1(a) := \begin{pmatrix} 0 & 0 & \sqrt{-1}\bar{a} \\ 0 & 0 & 0 \\ -\sqrt{-1}a & 0 & 0 \end{pmatrix},$$

$$A_3^1(a) := \begin{pmatrix} 0 & \sqrt{-1}a & 0 \\ -\sqrt{-1}\bar{a} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\tilde{A}_i^1(a) \in \mathrm{End}_{\mathbb{R}}(\mathcal{J}^1)$ is defined as

$$\tilde{A}_i^1(a) := [A_i^1(a), X] \quad \text{for } X \in \mathcal{J}^1$$

and the subspaces u_i^1 of $\text{End}_{\mathbb{R}}(\mathcal{J}^1)$ as $u_i^1 := \{\tilde{A}_i^1(a) \mid a \in \mathbf{O}\}$. The differential $d\tilde{\sigma}_i$ of the involutive automorphism $\tilde{\sigma}_i$ is written by same letter $\tilde{\sigma}_i$. Then $\tilde{\sigma}_i(\phi) = \sigma_i \phi \sigma_i$ for $\phi \in \mathfrak{f}_{4(-20)}$.

Lemma 4.1.

- (1) ([9], cf. [24, Proposition 2.1]) $\mathfrak{f}_{4(-20)} = \mathfrak{d}_4 \oplus u_1^1 \oplus u_2^1 \oplus u_3^1$.
- (2) ([43], cf. [26, Lemma 4.2]) $\tilde{\sigma}_1$ is a Cartan involution.
- (3) If $\mathfrak{f}_{4(-20)} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition with respect to $\tilde{\sigma}_1$, then

$$\mathfrak{k} = \mathfrak{d}_4 \oplus u_1^1, \quad \mathfrak{p} = u_2^1 \oplus u_3^1.$$

Now $\tilde{A}_3^1(1) \in \mathfrak{p}$. Let us define the abelian subspace \mathfrak{a} of \mathfrak{p} , the 1-parameter subgroup A , and $\alpha \in \mathfrak{a}^*$ as

$$\mathfrak{a} := \{t\tilde{A}_3^1(1) \mid t \in \mathbb{R}\}, \quad A := \{\exp(t\tilde{A}_3^1(1)) \mid t \in \mathbb{R}\}, \quad \alpha(\tilde{A}_3^1(1)) := 1$$

respectively. Denote

$$\begin{aligned} \mathfrak{g}_\lambda &:= \{\phi \in \mathfrak{f}_{4(-20)} \mid [H, \phi] = \lambda(H)\phi \text{ for all } H \in \mathfrak{a}\}, \\ \Sigma &:= \{\lambda \in \mathfrak{a}^* \mid \lambda \neq 0, \mathfrak{g}_\lambda \neq \{0\}\}, \end{aligned}$$

and the centralizer \mathfrak{a} of the group K and its Lie algebra as

$$\begin{aligned} M &:= Z_K(\mathfrak{a}) = \{k \in K \mid k\tilde{A}_3^1(1)k^{-1} = \tilde{A}_3^1(1)\}, \\ \mathfrak{m} &:= Z_{\mathfrak{k}}(\mathfrak{a}) = \{\phi \in \mathfrak{k} \mid [\phi, \tilde{A}_3^1(1)] = 0\} \end{aligned}$$

respectively. For $p \in \text{Im}\mathbf{O}$, $l_p, r_p, t_p \in \text{End}_{\mathbb{R}}(\mathbf{O})$ are defined by

$$l(p)x := px, \quad r(p)(x) := xp, \quad t(p)x := px + xp \quad \text{for } x \in \mathbf{O}$$

respectively. Then we see that

$$\delta(p) := d\varphi_0(l_p, r_p, t_p) \in \mathfrak{d}_4.$$

For $p \in \text{Im}\mathbf{O}$ and $x \in \mathbf{O}$, denote

$$\begin{aligned} \mathcal{G}_1(x) &:= \tilde{A}_1^1(x) + \tilde{A}_2^1(-\bar{x}), & \mathcal{G}_2(p) &:= -\tilde{A}_3^1(p) - \delta(p), \\ \mathcal{G}_{-1}(x) &:= \tilde{A}_1^1(x) + \tilde{A}_2^1(\bar{x}), & \mathcal{G}_{-2}(p) &:= \tilde{A}_3^1(p) - \delta(p) \end{aligned}$$

For $i = \pm 1$ and $j = \pm 2$, denote the subspaces \mathfrak{g}_i and \mathfrak{g}_j as $\mathfrak{f}_{4(-20)}$

$$\mathfrak{g}_i := \{\mathcal{G}_i(p) \mid p \in \text{Im}\mathbf{O}\}, \quad \mathfrak{g}_j := \{\mathcal{G}_j(x) \mid x \in \mathbf{O}\}$$

respectively.

Proposition 4.2. (cf. [26, Proposition 4.4])

$$\begin{aligned} M &= (F_{4(-20)})_{E_1, F_3^1(1)} = (F_{4(-20)})_{E_2, F_3^1(1)} = (F_{4(-20)})_{E_1, E_2, E_3, F_3^1(1)} \\ &= \varphi_0(\text{Spin}(7)). \end{aligned}$$

Lemma 4.3. (cf. [26, Lemma 4.5])

\mathfrak{a} is a maximal abelian subspace of \mathfrak{p} ,

$$\mathfrak{g}_{\pm\alpha} = \mathfrak{g}_{\pm 1}, \quad \mathfrak{g}_{\pm 2\alpha} = \mathfrak{g}_{\pm 2} \quad (resp),$$

and $(\mathfrak{f}_{4(-20)}, \mathfrak{a})$ -root space decomposition of $\mathfrak{f}_{4(-20)}$ is given by

$$\mathfrak{f}_{4(-20)} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{2\alpha} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

So the nilpotent subalgebras \mathfrak{n}^{\pm} are defined as

$$\mathfrak{n}^+ := \mathfrak{g}_{2\alpha} \oplus \mathfrak{g}_{\alpha} = \{\mathcal{G}_2(p) + \mathcal{G}_1(x) \mid p \in \text{Im}\mathbf{O}, x \in \mathbf{O}\},$$

$$\mathfrak{n}^- := \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} = \{\mathcal{G}_{-2}(p) + \mathcal{G}_{-1}(x) \mid p \in \text{Im}\mathbf{O}, x \in \mathbf{O}\} \quad (resp).$$

Then

$$[\mathfrak{n}^+, [\mathfrak{n}^+, \mathfrak{n}^+]] = [\mathfrak{n}^-, [\mathfrak{n}^-, \mathfrak{n}^-]] = 0.$$

And the nilpotent subgroups N^{\pm} of $F_{4(-20)}$ are defined as

$$N^+ := \exp \mathfrak{n}^+ = \{\exp(\mathcal{G}_2(p) + \mathcal{G}_1(x)) \mid p \in \text{Im}\mathbf{O}, x \in \mathbf{O}\},$$

$$N^- := \exp \mathfrak{n}^- = \{\exp(\mathcal{G}_{-2}(p) + \mathcal{G}_{-1}(x)) \mid p \in \text{Im}\mathbf{O}, x \in \mathbf{O}\} \quad (resp).$$

Lemma 4.4.

$$(1) \exp \mathcal{G}_2(p) \exp \mathcal{G}_1(x) = \exp(\mathcal{G}_2(p) + \mathcal{G}_1(x)) = \exp \mathcal{G}_1(x) \exp \mathcal{G}_2(p).$$

$$(2) \tilde{\sigma}_1 \mathfrak{n}^+ = \mathfrak{n}^- \text{ and } \tilde{\sigma}_1 \mathfrak{n}^- = \mathfrak{n}^+. \text{ Furthermore,}$$

$$\tilde{\sigma}_1(\mathcal{G}_{\pm 2}(p) + \mathcal{G}_{\pm 1}(x)) = \mathcal{G}_{\mp 2}(p) + \mathcal{G}_{\mp 1}(x) \quad (resp).$$

$$(3) \tilde{\sigma}_1(N^+) = N^- \text{ and } \tilde{\sigma}_1(N^-) = N^+. \text{ Furthermore,}$$

$$\tilde{\sigma}_1(\exp(\mathcal{G}_{\pm 2}(p) + \mathcal{G}_{\pm 1}(p))) = \exp(\mathcal{G}_{\mp 2}(p) + \mathcal{G}_{\mp 1}(p)) \quad (resp).$$

Lemma 4.5. ([26, Lemma 5.3])

Let $g = (g_1, g_2, g_3), h \in \text{Spin}(7), p, q \in \text{Im}\mathbf{O}, x, y \in \mathbf{O}$.

$$\begin{aligned} & \exp(\mathcal{G}_2(p) + \mathcal{G}_1(x))\varphi_0(g) \exp(\mathcal{G}_2(q) + \mathcal{G}_1(y))\varphi_0(h) \\ &= \exp(\mathcal{G}_2(p + g_3q + \text{Im}(x\overline{g_1y})) + \mathcal{G}_1(x + g_1y))\varphi_0(gh). \end{aligned}$$

Let us consider $G := \text{Spin}(7) \times \text{Im}\mathbf{O} \times \mathbf{O}$ in which multiplication is defined by

$$(g, p, x)(h, q, y) := (gh, p + g_3q + \text{Im}(x\overline{g_1y}), x + g_1y)$$

where $p, q \in \text{Im}\mathbf{O}, x, y \in \mathbf{O}$ and $g = (g_1, g_2, g_3), h \in \text{Spin}(7)$. Denote

$$H := \{(g, 0, 0) \mid g \in \text{Spin}(7)\},$$

$$N := \{(1, p, x) \mid p \in \text{Im}\mathbf{O}, x \in \mathbf{O}\},$$

$$G' := \{(g, p, 0) \mid g \in \text{Spin}(7), p \in \text{Im}\mathbf{O}\}, \quad N_1 := \{(1, p, 0) \mid p \in \text{Im}\mathbf{O}\},$$

$$G'' := \{(g, p, q) \mid g \in G_2, p, q \in \text{Im}\mathbf{O}\},$$

$$H'' := \{(g, 0, 0) \mid g \in G_2\},$$

$$N_2 := \{(1, p, q) \mid p, q \in \text{Im}\mathbf{O}\}.$$

Lemma 4.6. ([26, Lemma 5.2])

- (1) G is a group with respect to the multiplication.
- (2) H, N, G', N_1, G'', H'' and N_2 are subgroups of G .
- (3) We have

$$G = H \ltimes N, \quad G' = H \ltimes N_1, \quad G'' = H'' \ltimes N_2.$$

Denote $\text{Spin}(7) := H$, $\text{Im}\mathbf{O} \times \mathbf{O} := N$, $\text{Im}\mathbf{O} = N_1$, $G_2 := H''$ and $\text{Im}\mathbf{O} \times \text{Im}\mathbf{O} := N''$ so that

$$\begin{aligned} \text{Spin}(7) \ltimes (\text{Im}\mathbf{O} \times \mathbf{O}) &= G, & \text{Spin}(7) \ltimes \text{Im}\mathbf{O} &= G' \\ G_2 \ltimes (\text{Im}\mathbf{O} \times \text{Im}\mathbf{O}) &= G''. \end{aligned}$$

The homomorphisms $\varphi : \text{Spin}(7) \ltimes (\text{Im}\mathbf{O} \times \mathbf{O}) \rightarrow (F_{4(-20)})_{P^-}$, $\varphi_1 : \text{Spin}(7) \ltimes \text{Im}\mathbf{O} \rightarrow (F_{4(-20)})_{E_3, P^-}$ and $\varphi_2 : G_2 \ltimes (\text{Im}\mathbf{O} \times \text{Im}\mathbf{O}) \rightarrow (F_{4(-20)})_Q$ are defined as

$$\begin{aligned} \varphi(g, p, x) &= \exp(\mathcal{G}_2(p) + \mathcal{G}_1(x))\varphi(g), & \varphi_1(g, p) &= \exp(\mathcal{G}_2(p))\varphi(g) \\ \varphi_2(g, p, q) &= \exp(\mathcal{G}_2(p) + \mathcal{G}_1(q))\varphi(g) & \text{for } p, q \in \text{Im}\mathbf{O} \text{ and } x \in \mathbf{O} \end{aligned}$$

respectively.

Proposition 4.7. ([26, Proposition 5.6])

- (1) φ_1 is an isomorphism onto $(F_{4(-20)})_{E_3, P^-}$.
- (2) φ is an isomorphism onto $(F_{4(-20)})_{P^-}$.
- (3) φ_2 is an isomorphism onto $(F_{4(-20)})_Q$.

The key of proof of (2): By direct calculation,

$$\text{Orb}_{N^+}(E_3) = \{X \in \mathcal{J}^1 \mid P^- \times X = -\frac{1}{2}P^-, X^{\times 2} = 0, \text{tr}(X) = 1\}.$$

Then this equation deduces $\text{Orb}_{N^+}(E_3) = \text{Orb}_{(F_{4(-20)})_{P^-}}(E_3)$. \square

The mappings $\psi_1 : F_{4(-20)} \rightarrow \mathbf{O}$, $\psi_2 : F_{4(-20)} \rightarrow \text{Im}\mathbf{O}$ and $\psi_3 : F_{4(-20)} \rightarrow F_{4(-20)}$ are defined as for $g \in F_{4(-20)}$,

$$\begin{aligned} \psi_1(g) &:= \frac{1}{2}((gE_3)_{F_1^1} + \overline{(gE_3)_{F_2^1}}), \\ \psi_2(g) &:= -\frac{1}{2}\text{Im}\left((g(-E_1 + E_2))_{F_3^1}\right), \\ \psi_3(g) &:= \exp(-\mathcal{G}_1(\psi_1(g)) - \mathcal{G}_2(\psi_2(g)))g \end{aligned}$$

respectively.

Proposition 4.8. ([26, Proposition 5.7])

- (1) Let $g \in (F_{4(-20)})_{P^-}$. Then $\psi_3(g) \in M$ and

$$g = \exp(\mathcal{G}_1(\psi_1(g)) + \mathcal{G}_2(\psi_2(g)))\psi_3(g) \in N^+M.$$

- (2) We have

$$(F_{4(-20)})_{P^-} = N^+M = MN^+.$$

5. THE ORBIT TYPES OF $F_{4(-20)}$ -ORBITS ON \mathcal{J}^1 .

Main Theorem 2. (The orbit types of $F_{4(-20)}$ -orbits on \mathcal{J}^1 [26, Main Theorem 1])

The orbit types of $F_{4(-20)}$ -orbits on \mathcal{J}^1 are given as follows.

(1) Assume that $X \in \mathcal{J}^1$ admits the characteristic roots $\lambda_1 > \lambda_2 > \lambda_3$. Then X can be transformed to the following canonical forms by $F_{4(-20)}$ with the following type of stabilizer group.

The canonical forms of X	The type of stabilizer group
1. $\text{diag}(\lambda_1, \lambda_2, \lambda_3)$	$\text{Spin}(8)$
2. $\text{diag}(\lambda_2, \lambda_3, \lambda_1)$	$\text{Spin}(8)$
3. $\text{diag}(\lambda_3, \lambda_1, \lambda_2)$	$\text{Spin}(8)$

(2) Assume that $X \in \mathcal{J}^1$ admits the characteristic roots $\lambda_1 \in \mathbb{R}$, $p \pm \sqrt{-1}q$ with $p \in \mathbb{R}$ and $q > 0$. Then X can be transformed to the following canonical form by $F_{4(-20)}$ with the following type of stabilizer group.

The canonical forms of X	The type of stabilizer group
4. $\text{diag}(p, p, \lambda_1) + F_3^1(q)$	$\text{Spin}^0(7, 1)$

(3) Assume that $X \in \mathcal{J}^1$ admits the characteristic roots λ_1 of multiplicity 1 and λ_2 of multiplicity 2. Then X can be transformed to the following canonical forms by $F_{4(-20)}$ with the following types of stabilizer group.

The canonical forms of X	The type of stabilizer group
5. $\text{diag}(\lambda_1, \lambda_2, \lambda_2)$	$\text{Spin}(9)$
6. $\text{diag}(\lambda_2, \lambda_2, \lambda_1)$	$\text{Spin}^0(8, 1)$
7. $\text{diag}(\lambda_2, \lambda_2, \lambda_1) + P^+$	$\text{Spin}(7) \ltimes \text{ImO}$
8. $\text{diag}(\lambda_2, \lambda_2, \lambda_1) + P^-$	$\text{Spin}(7) \ltimes \text{ImO}$

(4) Assume that $X \in \mathcal{J}^1$ admits the characteristic root of multiplicity 3. Then X can be transformed to the following canonical forms by $F_{4(-20)}$ with the following types of stabilizer group.

The canonical forms of X	The type of stabilizer group
9. $\frac{1}{3}\text{tr}(X)E$	$F_{4(-20)}$
10. $\frac{1}{3}\text{tr}(X)E + P^+$	$\text{Spin}(7) \ltimes (\text{ImO} \times \text{O})$
11. $\frac{1}{3}\text{tr}(X)E + P^-$	$\text{Spin}(7) \ltimes (\text{ImO} \times \text{O})$
12. $\frac{1}{3}\text{tr}(X)E + Q^+(1)$	$G_2 \ltimes (\text{ImO} \times \text{ImO})$

6. THE THREE DECOMPOSITIONS OF A LINEAR CONNECTED SEMISIMPLE NONCOMPACT LIE GROUPS.

Let G be a linear connected semisimple Lie group with its Lie algebra \mathfrak{g} over \mathbb{R} . Let θ be a Cartan involution of \mathfrak{g} , $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition, \mathfrak{a} is a maximal abelian subspace of \mathfrak{p} , $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$. \mathfrak{a}^*

denotes the dual space of \mathfrak{a} . For any element $\lambda \in \mathfrak{a}^*$, let $\mathfrak{g}_\lambda := \{X \in \mathfrak{g} \mid [H, X] = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}$. λ is called a *root* of $(\mathfrak{g}, \mathfrak{a})$ if $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq \{0\}$. The set of roots of $(\mathfrak{g}, \mathfrak{a})$ is denoted by Σ . Then $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \sum_{\lambda \in \Sigma} \mathfrak{g}_\lambda$ follows. Denote by Σ^+ a set of positive root of $(\mathfrak{g}, \mathfrak{a})$ with respect to the some ordering in \mathfrak{a}^* , $\Sigma^- := \{-\lambda \mid \lambda \in \Sigma^+\}$, $\mathfrak{n}^+ := \sum_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$ and $\mathfrak{n}^- := \sum_{\lambda \in \Sigma^-} \mathfrak{g}_\lambda$. Then \mathfrak{n}^+ and \mathfrak{n}^- are nilpotent subalgebras, $\theta \mathfrak{n}^\pm = \mathfrak{n}^\mp$ (*resp*), and $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n}^+$ follow. Suppose that there exists an involutive automorphism Θ on G such that the differential $d\Theta = \theta$, and the center $Z(G)$ of G is finite. Denote the subgroup $K := G^\Theta$ of G . Then $\text{Lie}(K) = \mathfrak{k}$ and K is connected, closed, and K is a maximal compact subgroup of G since $Z(G)$ of G is finite. Set $A := \exp \mathfrak{a}$, $M := Z_K(\mathfrak{a}) = \{k \in K \mid kXk^{-1} = X \text{ for all } X \in \mathfrak{a}\}$ and $N^\pm := \exp \mathfrak{n}^\pm$ (*resp*). Then the identity connected component M^0 of M is the analytic subgroup corresponding to \mathfrak{m} , and $\Theta N^\pm = N^\mp$ (*resp*). denote the normalizer of \mathfrak{a} of the group K as $M^* := N_K(\mathfrak{a}) = \{k \in K \mid kak^{-1} \subset \mathfrak{a}\}$ and the finite factor group $W := M^*/M$.

For all $w \in W$, we fix a representative $\bar{w} \in M^*$. Then the following decompositions:

$$(1) \ G = \coprod_{w \in W} MAN^+ \bar{w} N^- \quad (\text{Bruhat decomposition}),$$

$$(1)' \ G = \overline{MAN^+ N^-} \quad (\text{Gauss decomposition}),$$

$$(2) \ G = KAN^+ \quad (\text{Iwasawa decomposition}).$$

(cf. [15], [18], [27], [23]). In (1)', the set $MAN^+ N^-$ is open dense in G , and so almost any $g \in G$ can be expressed by

$$g = m_G(g) a_G(g) n_G(g) \bar{n}_G(g)$$

for some $m_G(g) \in M$, $a_G(g) \in A$, $n_G(g) \in N^+$ and $\bar{n}_G(g) \in N^-$ with uniquely determined factors. In (2), any $g \in G$ can be uniquely expressed by

$$g = k(g)(\exp H(g))n(g)$$

for some $k(g) \in K$, $H(g) \in \mathfrak{a}$ and $n(g) \in N$.

A *signature of roots* is defined by the mapping ϵ of Σ to $\{-1, 1\}$ such that ϵ satisfies the conditions:

- (i) $\epsilon(\lambda) = \epsilon(-\lambda)$ for all $\lambda \in \Sigma$,
- (ii) $\epsilon(\lambda + \mu) = \epsilon(\lambda)\epsilon(\mu)$ if $\lambda, \mu, \lambda + \mu \in \Sigma$

[27, Definition 1.1]. For the Cartan involution θ and any signature ϵ of roots, let us define an involutive automorphism θ_ϵ of \mathfrak{g} such that

- (i) $\theta_\epsilon(X) := \epsilon(\lambda)\theta(X)$ for all $\lambda \in \Sigma$ and $X \in \mathfrak{g}_\lambda$,
- (ii) $\theta_\epsilon(X) := \theta(X)$ for all $X \in \mathfrak{a} \oplus \mathfrak{m}$

[27, Definition 1.2]. θ_ϵ is called the (θ, ϵ) -*involution* of \mathfrak{g} . Set

$$\mathfrak{k}_\epsilon := \{X \in \mathfrak{g} \mid \theta_\epsilon X = X\}, \quad \mathfrak{p}_\epsilon := \{X \in \mathfrak{g} \mid \theta_\epsilon X = -X\}.$$

Then $\mathfrak{g} = \mathfrak{k}_\epsilon \oplus \mathfrak{p}_\epsilon$. Let $(K_\epsilon)_0$ be the analytic subgroup of G with the Lie algebra \mathfrak{k}_ϵ and the subgroup K_ϵ of G as $K_\epsilon := (K_\epsilon)_0 M$. In fact, since all elements of M normalize $(K_\epsilon)_0$ by [27, Lemma 1.4(i)], K_ϵ is a subgroup of G . Denote $M_\epsilon^* := K_\epsilon \cap M^*$ and $W_\epsilon := M_\epsilon^*/M$.

Proposition 6.1. (**Iwasawa decomposition with respect to K_ϵ in sense of T. Oshima and J. Sekiguchi** [27, Proposition 1.10])
Let the factor set $W_\epsilon \backslash W = \{w_1 = 1, w_2, \dots, w_r\}$ where $r = [W : W_\epsilon]$. Fix representatives $\bar{w}_1 = 1, \bar{w}_2, \dots, \bar{w}_r \in M_\epsilon^ = K_\epsilon \cap M^*$ for $w_1 = 1, w_2, \dots, w_r$. Then the decomposition*

$$G \supset \cup_{i=1}^r K_\epsilon \bar{w}_i A N^+$$

has the following properties.

- (1) *If $k\bar{w}_i a n = k'\bar{w}_j a' n'$ with $k, k' \in K_\epsilon$, $a, a' \in A$ and $n, n' \in N^+$, then $k = k'$, $i = j$, $a = a'$ and $n = n'$.*
- (2) *The map $(k, a, n) \mapsto k\bar{w}_i a n$ defines an analytic diffeomorphism of the product manifold $K_\epsilon \times A \times N^+$ onto the open submanifold $K_\epsilon \bar{w}_i A N^+$ of G ($i = 1, \dots, r$).*
- (3) *The submanifolds $\cup_{i=1}^r K_\epsilon \bar{w}_i A N^+$ is open dense in G .*

7. THE GAUSS DECOMPOSITION OF $F_{4(-20)}$.

We have

$$\mathcal{N}_1^-(\mathbf{O}) = \text{Orb}_{F_{4(-20)}}(P^-) \simeq F_{4(-20)} / (F_{4(-20)})_{P^-} = F_{4(-20)} / N^+ M.$$

So considering AN^- -orbits on $\mathcal{N}_1^-(\mathbf{O})$, we obtain:

Main Theorem 3. (The Bruhat and Gauss decomposition of $F_{4(-20)}$ [26, Main Theorem 2])

- (1) *Assume that $g \in F_{4(-20)}$ and $(gP^+ | P^-) \neq 0$. Let*

$$\begin{aligned} t &:= -\frac{1}{2} \log \left(-\frac{(gP^+ | P^-)}{4} \right) \in \mathbb{R}, \\ a_G(g) &:= \exp(t\tilde{A}_3^1(1)) \in A, \\ \bar{n}_G(g) &= \tilde{\sigma}_1 \left(\exp(-\mathcal{G}_1 \left(\frac{(\sigma_1 g^{-1} P^-)_{F_1^1} - \overline{(\sigma_1 g^{-1} P^-)_{F_2^1}}}{(gP^+ | P^-)} \right) \right. \\ &\quad \left. - \mathcal{G}_2 \left(\frac{\text{Im}((\sigma_1 g^{-1} P^-)_{F_3^1})}{(gP^+ | P^-)} \right) \right) \in N^-, \\ n_G(g) &:= \exp(t(\mathcal{G}_1(\psi_1(a_G(g)\bar{n}_G(g)g^{-1}) \\ &\quad + 2\mathcal{G}_2(\psi_2(a_G(g)\bar{n}_G(g)g^{-1}))) \in N^+, \\ m_G(g) &:= \psi_3(a_G(g)\bar{n}_G(g)g^{-1})^{-1}. \end{aligned}$$

Then

(i) $(gP^+|P^-) < 0$, and $a_G(g)$, $\bar{n}_G(g)$, $n_G(g)$, $m_G(g)$ are well-defined,

(ii) $m_G(g) \in M$ and

$$g = m_G(g)a_G(g)n_G(g)\bar{n}_G(g) \in MAN^+N^-.$$

(2) Assume $g \in F_{4(-20)}$ and $(gP^+|P^-) = 0$. Let

$$t := -\frac{1}{2} \log(-(gE_1|P^-)) \in \mathbb{R},$$

$$a'(g) = \exp(t\tilde{A}_3^1(1)) \in A,$$

$$n'(g) := \exp(t(\mathcal{G}_1(\psi_1(\sigma_1 a'(g)g^{-1}) + 2\mathcal{G}_2(\psi_2(\sigma_1 a'(g)g^{-1})))) \in N^+,$$

$$m'(g) := \psi_3(\sigma_1 a'(g)g^{-1})^{-1}.$$

Then

(i) $(gE_1|P^-) < 0$, and $a'(g)$, $n'(g)$, $m'(g)$ are well-defined,

(ii) $m'(g) \in M$ and

$$g = m'(g)a'(g)n'(g)\sigma_1 \in MAN^+\sigma_1 = MAN^+\sigma_1 N^-.$$

(3) The following equations hold.

$$\begin{aligned} MAN^+N^- &= \{g \in F_{4(-20)} \mid (gP^+|P^-) \neq 0\} \\ &= \{g \in F_{4(-20)} \mid (gP^+|P^-) < 0\} \neq \emptyset, \\ MAN^+\sigma_1 &= MAN^+\sigma_1 N^- \\ &= \{g \in F_{4(-20)} \mid (gP^+|P^-) = 0\} \neq \emptyset. \end{aligned}$$

Epecially,

$$\begin{aligned} F_{4(-20)} &= MAN^+N^- \coprod MAN^+\sigma_1 N^- \quad (\text{Bruhat decomposition}) \\ &= MAN^+N^- \coprod MAN^+\sigma_1 \end{aligned}$$

(4) MAN^+N^- is open dense in $F_{4(-20)}$. *Epecially*

$$F_{4(-20)} = \overline{MAN^+N^-} \quad (\text{Gauss decomposition}).$$

8. THE IWASAWA DECOMPOSITION OF $F_{4(-20)}$.

We have

$$\mathcal{H}(\mathbf{O}) = \text{Orb}_{F_{4(-20)}}(E_1) \simeq F_{4(-20)} / (F_{4(-20)})_{E_1} = F_{4(-20)} / K.$$

So considering AN^+ -orbits on $\mathcal{H}(\mathbf{O})$, we obtain:

Main Theorem 4. (The Iwasawa decomposition of $F_{4(-20)}$ [26, Main Theorem 3])

For any $g \in F_{4(-20)}$, let

$$H(g) := \frac{1}{2} \log(-(gP^-|E_1)) \tilde{A}_3^1(1) \in \mathfrak{a},$$

$$n(g) := \exp(\mathcal{G}_1 \left(\frac{(g^{-1}E_1)_{F_1^1} - \overline{(g^{-1}E_1)_{F_2^1}}}{(gP^-|E_1)} \right) + \mathcal{G}_2 \left(\frac{\text{Im}((g^{-1}E_1)_{F_3^1})}{(gP^-|E_1)} \right)) \\ \in N^+$$

$$k(g) := gn(g)^{-1} \exp(-H(g)).$$

Then

(1) $(gP^-|E_1) < 0$. Especially $H(g)$, $n(g)$ and $k(g)$ is well-defined.

(2) $k(g) \in K$ and

$$g = k(g)(\exp H(g))n(g) \in KAN^+.$$

9. THE IWASAWA DECOMPOSITION WITH RESPECT TO K_ϵ .

For $G = F_{4(-20)}$, let ϵ be a signature of root defined by

$$\epsilon(\alpha) = \epsilon(-\alpha) := -1, \quad \epsilon(2\alpha) = \epsilon(-2\alpha) := 1.$$

Denote the $(\tilde{\sigma}_1, \epsilon)$ -involution by $(\tilde{\sigma}_1)_\epsilon$, and use same notations \mathfrak{k}_ϵ , $(K_\epsilon)_0$, K_ϵ , M^* , M_ϵ^* , W and W_ϵ corresponding to notations of general G respectively.

Proposition 9.1. ([26, Lemma 6.2])

- (1) $(\tilde{\sigma}_1)_\epsilon = \tilde{\sigma}_2$.
- (2) $K_\epsilon = (F_{4(-20)})_{E_2}$.
- (3) $M^* = M \amalg \sigma_1 M$. Especially $W = \{M, \sigma_1 M\} \cong \mathbb{Z}_2$.
- (4) $M_\epsilon^* = M \amalg \sigma_1 M$. Especially $W_\epsilon = \{M, \sigma_1 M\}$ and $[W : W_\epsilon] = 1$.

We have

$$\mathcal{H}'(\mathbf{O}) = \text{Orb}_{F_{4(-20)}}(E_2) \simeq F_{4(-20)} / (F_{4(-20)})_{E_2} = F_{4(-20)} / K_\epsilon.$$

So considering AN^+ -orbits on $\mathcal{H}'(\mathbf{O})$, we obtain:

Main Theorem 5. (The Iwasawa decomposition with respect to K_ϵ [26, Main Theorem 4])

Let \mathcal{D} be the domain of $F_{4(-20)}$ defined by

$$\mathcal{D} := \{g \in F_{4(-20)} \mid (gP^-|E_2) > 0\}.$$

For any $g \in \mathcal{D}$, let

$$\begin{aligned} H_\epsilon(g) &:= \frac{1}{2} \log((gP^-|E_2)) \tilde{A}_3^1(1) \in \mathfrak{a}, \\ n_\epsilon(g) &:= \exp(\mathcal{G}_1 \left(\frac{(g^{-1}E_2)_{F_1^1} - \overline{(g^{-1}E_2)_{F_2^1}}}{(gP^-|E_2)} \right) \\ &\quad + \mathcal{G}_2 \left(\frac{\text{Im}((g^{-1}E_2)_{F_3^1})}{(gP^-|E_2)} \right)) \in N^+ \\ k_\epsilon(g) &:= gn_\epsilon(g)^{-1} \exp(-H_\epsilon(g)). \end{aligned}$$

Then

(1) $k_\epsilon(g) \in K_\epsilon$ and

$$g = k_\epsilon(g)(\exp H_\epsilon(g))n_\epsilon(g) \in K_\epsilon AN^+.$$

(2) $\mathcal{D} = K_\epsilon AN^+ = \{g \in F_{4(-20)} \mid (gP^-|E_2) \neq 0\}$. Furthermore, \mathcal{D} is open dense in $F_{4(-20)}$.

Moreover we have:

Theorem 9.2. ([26, Theorem 9.6])

(1) The following equations hold.

$$\begin{aligned} K_\epsilon MAN^+ &= \{g \in F_{4(-20)} \mid (gP^-|E_2) \neq 0\} \\ &= \{g \in F_{4(-20)} \mid (gP^-|E_2) > 0\} = K_\epsilon AN^+. \end{aligned}$$

$$(2) K_\epsilon \exp\left(-\tilde{A}_1^1\left(\frac{\pi}{2}\right)\right) MAN^+ = \{g \in F_{4(-20)} \mid (gP^-|E_2) = 0\}.$$

$$(3) F_{4(-20)} = K_\epsilon MAN^+ \amalg K_\epsilon \exp\left(-\tilde{A}_1^1\left(\frac{\pi}{2}\right)\right) MAN^+.$$

Remark 9.3. Theorem 9.2(3) is a special case of [21, Theorems 3], so the decomposition in Theorem 9.2(3) is called a *Matsuki decomposition* of $F_{4(-20)}$.

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